1. Introduction

The maintenance Problem of a multistate degenerative system under the bivariate replacement policy has (k+l) possible states, where the states 1,2,…k denotes, respectively, the first-type working state, the second-type working state,…kth type working state and the states (k+1), (k+2),…,(k+l) denote, respectively, the first-type failure state, the second-type failure state,… and the lth failure state of the system. The occurrences of these types of failures are stochastic and mutually exclusive.

2.2 Whenever the system fails in any of the failure states, it will be repaired. The system will be replaced by an identical one some times later.

2.3 Let Xn be the survival time of the system after (n−1)th repair. then \( \{X_n, n=1,2,\ldots\} \) forms a non-increasing geometric process with parameter \( p > 1 \) and
2.4 Let \( Y_n \) be the repair time of the system after \( n \)-th failure. Then \( \{Y_n, n = 1, 2, \ldots\} \) forms a non-decreasing geometric process with parameter \( b, 0 < b < 1 \) and \( E(Y_i) = \mu \geq 0 \). Here \( \mu = 0 \) mean that repair time is negligible.

2.5 If the system in working state \( i \) is operating, then let the reward rate be \( r_i \) if the system in failure state \( (k = i) \) is under repair, the repair cost is \( c_i \). The replacement cost comprises two parts. One part is the replacement cost \( R \) and the other proportional to the replacement time \( Z \) at rate \( C_p \). In other words, the replacement cost is given by \( R + C_p Z \).

2.6 Assume that \( 1 \leq a_1 < a_2 < \ldots < a_k \) and \( 1 \leq b_1 \geq b_2 \geq \ldots \geq b_k > 0 \).

2.7 Assume that \( F_n(t) \) is the cumulative distribution of \( L_n = \sum_{i=1}^{n} X_i \) and \( G_n(t) \) be the cumulative distribution of \( M_n = \sum_{i=1}^{n} Y_i \).

2.8 The survival time \( X_n \), the repair time \( Y_n \) and the replacement time \( Z \), \((n = 1, 2, \ldots, k)\) are independent random variables. We now describe the probability structure of the model.

Assume that the transition probability from working state \( i, i = 1, 2, \ldots, k \) to failure state \( (k + j), j = 1, 2, \ldots, l \) is
\[
P(S(s_{n+1}) = k + j| S(t_n) = i) = q_{ij},
\]
with \( \sum_{j=1}^{l} q_{ij} = 1 \). Moreover, the transition probability from failure state \( k + j, j = 1, 2, \ldots, l \) to working state \( i, i = 1, 2, \ldots, k \) is given by
\[
P(S(t) = i| S(s_n) = k + j) = P_i
\]
with \( \sum_{i=1}^{k} P_i = 1 \).

Assume that there exists a life-time distribution \( U(t) \) and \( a_i > 0, i = 1, 2, \ldots, k \) such that
\[
P(X_i \leq t) = U(t)
\]
and
\[
P(X_2 \leq t| S(t_1) = i) = U(a_i t), i = 1, 2, \ldots, k
\]
where \( 1 \leq a_1 < a_2 < \ldots < a_k \).

In general for \( i_j \in \{1, 2, \ldots, k\} \)
\[
P(X_n \leq t| S(t_{i-1}) = i_{i-1}, S(t_{i-1}) = i_{n-1}) = U(a_{i_1}, \ldots, a_{i_n} t)
\]
Similarly, assume that there exists a life-time distribution \( V(t) \) and \( b_i > 0 \),
\[
i = 1, 2, \ldots, l
\]
such that
\[
P(Y_i \leq t| S(s_i) = k + i) = V(b_i t)
\]
where \( 1 \geq b_1 \geq b_2 \geq \ldots \geq b_l > 0 \) and in general, for \( i_j \in \{1, 2, \ldots, l\} \)
\[
P(Y_n \leq t| S(s_n) = k + i_1, \ldots, S(s_n) = k + i_n) = V(b_1, \ldots, b_l t)
\]

3. The Policy \((T, N)\)
In this section, we introduce and study a bivariate replacement policy \((T, N)\) for the multistate degenerative system, under which system is replaced at working age \( T \) or at the time of \( N \)-th failure, whichever occurs first. The problem is to choose an optimal replacement policy \((T, N)^*\) such that the long-run average cost per unit time is minimized. The working age \( T \) of the system at time is the commutation life-time given by
\[
T(t) = \begin{cases} t - M_n, & \text{if } L_n + M_n \leq t < L_{n+1} + M_n \\ L_{n+1}, & \text{if } L_{n+1} + M_n \leq t < L_{n+1} + M_{n+1} \end{cases}
\]
where \( L_n = \sum_{i=1}^{n} X_i \) and \( M_n = \sum_{i=1}^{n} Y_i \) and \( L_0 = M_0 = 0 \).

Following Lam [2005], the distribution of the survival time \( X_n \), in assumption 2.3 and the distribution of the repair time \( Y_n \) in assumption 2.4 are given by
\[
P(X_n \leq t) = \sum_{i=1}^{k} \frac{(n-1)!}{j_1! \ldots j_k!} U(a_1^{j_1} \ldots a_k^{j_k} t)
\]
\[
\sum_{i=1}^{k} p_i^{j_i} \left( \sum_{i=1}^{k} p_i^{j_i} t \right)
\]
where \( j_1, j_2, \ldots, j_k \in \mathbb{Z}^+ \) and
\[
P(Y_n \leq t) = \sum_{i=1}^{n} \frac{n!}{j_1! \ldots j_l!} q_1^{j_1} \ldots q_l^{j_l} V(b_1^{j_1} \ldots b_l^{j_l} t)
\]
where \( j_1, j_2, \ldots, j_l \in \mathbb{Z}^+ \), if \( E(X_i) = \lambda \), then the mean survival time is \( E(X_n) = \lambda \alpha^{n-1} \), for \( n > 1 \),
where \( \alpha = \frac{\sum_{i=1}^{k} p_i}{a_1} \) and if \( E(Y_i) = \mu \) then the mean repair time is \( E(Y_n) = \beta^n \mu \)
for \( n > 1 \), where
\[ \beta = \left[ \sum_{i=1}^{k} \frac{q_i}{b_i} \right] \]

Further if \( R_n = r_i \) denotes the reward earned after the \( n^{th} \) repair, where \( S(s_{n-1}) = i, i = 1, 2, \ldots, k \) then mean reward earned after \((n-1)^{th}\) repair is \( E(R_nX_n) = r\alpha^{n-1} \) and for \( n \geq 2 \) then expected reward after installation is given by
\[
E(R_nX_n) = r\alpha^{n-1}
\]

(3.3)

where \( r = \sum_{i=1}^{k} \frac{r_ip_i}{a_i} \) and if \( C_n = c_i \) denote the repair cost after the \( n^{th} \) failure, where \( S(s_n) = k+i, i = 1, 2, \ldots \) then mean repair cost after \( n^{th} \) failure is
\[
E(C_nX_n) = C\mu\beta^n
\]

(3.4)

\[ \text{Proof. Consider} \]
\[ E(w) = E\left[ \left( \sum_{i=1}^{N} Y_i \right) \chi(t_N \leq T) \right] + E\left[ \left( \sum_{i=1}^{N} Y_i \right) \chi(t_N > T) \right] + E(Z) \]
\[ = E\left[ \left( \sum_{i=1}^{N} Y_i \right) \chi(t_N \leq T) \chi(t_N > T) \right] + E(T\chi(t_N > T)) \]
\[ = \int_0^T uF_N(u)du + \sum_{i=1}^{N-1} \mu \beta^{-1} F_i(T) + \tau \]
\[ = \int_0^T uF_N(u)du + \sum_{i=1}^{N-1} \mu \beta^{-1} F_i(T) + \tau \]
\[ = \int_0^T uF_N(u)du + \sum_{i=1}^{N-1} \mu \beta^{-1} F_i(T) + \tau \]
\[ = \sum_{i=1}^{N-1} \mu \beta^{-1} F_i(T) + \tau \]

(3.5)

\[ \text{Lemma 3.1} \quad \text{The mean length of a cycle under the policy} \quad (T, N) \quad \text{is} \]
\[ E(w) = \int_0^T uF_N(u)du + \sum_{i=1}^{N-1} \mu \beta^{-1} F_i(T) + \tau \]

\[ \text{Lemma 3.2} \quad \text{If} \quad L_N \leq T \quad \text{and} \quad n \geq 2, \quad \text{then the expected reward earned is} \]
\[ E\left[ \left( \sum_{n=2}^{N} R_nX_n \right) \chi(t_N \leq T) \right] = \sum_{n=2}^{N} \mu \beta^{-1} \int_0^T uF_N(u) \]

(3.6)

\[ \text{Proof. Consider} \]
\[ E\left( \sum_{n=2}^{N} R_nX_n \chi(t_N \leq T) \right) = E\left( E\left[ \left( \sum_{n=2}^{N} R_nX_n \chi(t_N \leq T) \right) \right] \right) \]

where \( C = \sum_{i=1}^{n} \frac{c_iq_i}{b_i} \)

The length of a cycle under the bivariate replacement policy \((T, N)\) is
\[ W = \left( \sum_{i=1}^{N} X_i + \sum_{i=1}^{N-1} Y_i \right) \chi(L_N \leq T) + \left( T + \sum_{i=1}^{N} Y_i \right) \chi(L_N > T) + Z \]

where \( \eta = 0, 1, 2, \ldots, N - 1 \) is the number of failures before the working age of the system exceeds \( T \) and \( \chi(.) \) denote the indicate functions.From Leung [2006], we have
\[ E[\chi(t_i < T \leq L_N)] = P(L_i \leq T \leq L_N) \]
\[ = F_i(T) - F_N(T) \]

\[ \text{Lemma 3.1} \quad \text{The mean length of a cycle under the policy} \quad (T, N) \quad \text{is} \]
\[ E(w) = \int_0^T uF_N(u)du + \sum_{i=1}^{N-1} \mu \beta^{-1} F_i(T) + \tau \]

(3.5)
which on simplification yields (3.8)

\[ \int_0^T E \left( \sum_{n=2}^N R_n X_n / L_N \right) dF_N(u) \]
\[ = \sum_{n=2}^N r \lambda \alpha^{n-2} \int_0^T udF_N(u) \]
\[ = \left( 1 - \alpha^{N-1} \right) r \lambda \int_0^T udF_N(u) \]

which is (3.6)

**Lemma 3.3** If \( L_N > T \) and \( n \geq 2 \), then the expected reward earned is

\[ E \left[ \sum_{n=2}^N R_n X_n \right]_{(L_N > T)} = \sum_{n=2}^N r \lambda \alpha^{n-2} [F_n(T) - F_N(T)] \]

(3.7)

**Proof.** Consider

\[ E \left[ \sum_{n=2}^N R_n X_n \right]_{(L_N > T)} = E \left[ \sum_{n=2}^N R_n X_n \right]_{(L_n < T < L_N)} \]
\[ = \sum_{n=2}^N E(R_n X_n) E[\chi_{(L_n < T < L_N)}] \]
\[ = \sum_{n=2}^N r \lambda \alpha^{n-2} p[L_n < T < L_N] \]
\[ = \sum_{n=2}^N r \lambda \alpha^{n-2} [F_n(T) - F_N(T)] \]

which is (3.7)

**Lemma 3.4** If \( L_N \leq T \), then the expected repair cost is

\[ E \left[ \sum_{n=1}^{N-1} C_n Y_n \right]_{(L_N \leq T)} = \sum_{n=1}^{N-1} C_n \mu \beta^{n-1} F_N(T) \]

(3.8)

**Proof.** Consider

\[ E \left[ \sum_{n=1}^{N-1} C_n Y_n \right]_{(L_N \leq T)} = E \left[ E \left( \sum_{n=1}^{N-1} C_n Y_n / L_N \right) \chi_{(L_N \leq T)} \right] \]
\[ = \int_0^T E \left( \sum_{n=1}^{N-1} C_n Y_n / L_N \right) dF_N(u) \]
\[ = \int_0^T \sum_{n=1}^{N-1} E(C_n Y_n) dF_N(u) \]
\[ = \sum_{n=1}^{N-1} E(C_n Y_n) \int_0^T dF_N(u) \]
\[ = \sum_{n=1}^{N-1} C_n \mu \beta^{n-1} F_N(T) \]
\[ = c \mu \left( 1 + \beta + \beta^2 + \ldots + \beta^{N-2} \right) F_N(T) \]
\[ = c \mu \left( \frac{\beta^{N-1} - 1}{\beta - 1} \right) F_N(T) \]

which on simplification yields (3.8)

**Lemma 3.5** If \( L_N > T \), then the expected repair cost is
Here  \( C_{m}(T) \) is a bivariate function. Obviously, when \( N \) is fixed, \( C(T, N) \) is a function of \( T \) for fixed \( N = m \), it can be written as

\[
C(T, N) = C_{m}(T), \quad m = 1, 2, \ldots
\]

Thus for a fixed \( m \), we can fixed \( T_{m}^{*} \) by analytical or numerical methods such that \( C_{m}(T_{m}^{*}) \) is minimized. That is when \( N = 1, 2, \ldots, m, \ldots \) we can fixed
Because the total life time of a multistate degenerative system is limited, the minimum of the long run average cost per unit time exists. So we can determine the minimum of the long run average cost per unit time based on 

\[ C_1(T_1^*), C_2(T_2^*), \ldots, C_m(T_m^*) \] for example, if the minimum is denoted by \( C_n(T_n^*) \) we obtain the bivariate optimal replacement policy \( (T, N)^* \) such that

\[ C((T, N)^*) = \min_n C_n(T_n^*) \]

**CONCLUSION**

By concluding a repairable system for a monotone process model of a one component multistate degenerate system explicit expression for the long-run average cost per unit time under the bivariate replacement policy \( (T, N) \) is derived. Existence of optimality under the bivariate replacement policy is also deduced.

**References**


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