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Research Article

RELATIONS BETWEEN THE MAIN SCALARS OF A FIVE-DIMENSIONAL FINSLER SPACE AND ITS HYPERSURFACE

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A B S T R A C T

Gauree Shanker, G. C. Chaubey and Vinay Pandey [1] studied a five- dimensional Finsler space in terms of scalars with the help of 'Miron frame' which was discussed by M. Matsumoto and R. Miron [2]. On the other hand, the theory of hypersurface was discussed in detail by M. Matsumoto [3]. The purpose of the present paper is to obtain relation between the main scalars of a five-dimensional Finsler space and its hypersurface. For terms and notations, we refer to Matsumoto [4].

INTRODUCTION

Let us consider a five-dimensional Finsler space $F^5 = (M^5, L(x, y))$ whose fundamental metric function is $L(x, y)$. The normalized supporting element, metric tensor and Cartan tensor are defined by

$$l_i = \frac{\partial L}{\partial y^i}, \qquad g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\,\partial y^j}, \qquad C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$$

respectively.

A hypersurface M^4 of M^5 may be represented parametrically by the equations $x^i = x^i(u^\alpha)$, where u^α are the Gaussian co-ordinate on M^4 (Latin indices sum from 1 to 5, while Greek indices, except λ, μ, ν take values 1 to 4). We assume that the matrix consisting of projection factors $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ is of rank 4. Then $B_\alpha(u) = (B^i_\alpha(u))$ may be regarded as four independent vectors tangent to M^4 at the point $u = (u^\alpha)$ and a vector X^i tangent to M^4 at the point may be expressed uniquely in the form $X^i = B^i_\alpha X^\alpha$, where X^α are the components of the vectors with respect to the coordinate system (u^α).

To introduce a Finsler structure on M^4, the supporting element y^i is assumed to be tangent to M^4 at a point *u* of M^4, so that we may write

$$y^i = B^i_\alpha(u)\, v^\alpha. \tag{1}$$

Thus, $v = (v^\alpha)$ may be supposed as the supporting element of M^4 at the point *u*. Denoting y^i of (1.1) by $y^i(u, v)$, the function

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$$\underline{L}(u,v) = L(x(u), y(u,v)), \tag{2}$$

gives rise to a Finsler metric on M^4. Consequently, we get a four-dimensional Finsler space

$$F^4 = (M^4, \underline{L}(u,v))$$

where $\underline{L}$ is the induced metric function on F^4.

The induced metric function $\underline{L}(u,v)$ yields $l_\alpha = \frac{\partial \underline{L}}{\partial v^\alpha}$, the metric tensor $g_{\alpha\beta} = \frac{1}{2}\frac{\partial^2 L^2}{\partial v^\alpha \partial v^\beta}$ and the Cartan tensor $C_{\alpha\beta\gamma} = \frac{1}{2}.\frac{\partial g_{\alpha\beta}}{\partial v^\gamma}$ of F^4. Paying attention to $\frac{\partial B^i_\alpha}{\partial v^\beta} = 0$, from (2), we get

$$l_\alpha = l_i\, B^i_\alpha, \qquad g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta, \qquad C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma. \tag{3}$$

At each point *u* of F^4, a unit normal vector $B^i(u,v)$ is defined as

$$g_{ij} = B^i_\alpha\, B^j = 0, \qquad g_{ij} B^i\, B^j = 1. \tag{4}$$

The inverse projection factors $B^\alpha_i(u,v)$ of B^i_α is defined as

$$B^\alpha_i = g^{\alpha\beta} g_{ij}\, B^i_\beta, \tag{5}$$

where $g^{\alpha\beta}$ is the inverse tensor of the metric tensor $g_{\alpha\beta}$ of F^4.
From (5), it follows that

$$\begin{aligned} &(a)\quad B^i_\alpha\, B^\beta_i = \delta^\beta_\alpha, &&(b)\quad B^i_\alpha B_i = 0, \quad B^i B^\alpha_i = 0,\\ &(c)\quad B^i B_i = 1, &&(d)\quad B^i_\alpha\, B^\alpha_j + B^i B_j = \delta^i_j. \end{aligned} \tag{6}$$

Let us deduce the following tensors from the Cartan tensor C_{ijk}:

$$M_{\alpha\beta} = LC_{ijk} B^i_\alpha B^j_\beta B^k, \qquad M_\alpha = LC_{ijk} B^i_\alpha B^j B^k, \qquad M = LC_{ijk} B^i B^j B^k. \tag{7}$$

From (3), (6) and (7), we may write

$$\begin{aligned} LC_{ijk} B^j_\alpha B^k_\beta &= \underline{L} C_{\alpha\beta\gamma} B^\gamma_i + M_{\alpha\beta} B_i,\\ LC_{ijk} B^j_\alpha B^k &= M_{\alpha\beta} B^\beta_i + M_\alpha B_i,\\ LC_{ijk} B^j B^k &= M_\alpha B^\alpha_i + M B_i, \end{aligned} \tag{8}$$

which leads to

$$LC_i = (\underline{L} C_\alpha + M_\alpha)\, B^\alpha_i + (M + g^{\alpha\beta} M_{\alpha\beta})\, B_i, \tag{9}$$

where $C_i (= g^{jk} C_{ijk})$ and $C_\alpha (= g^{\beta\gamma} C_{\alpha\beta\gamma})$ are called torsion vectors of F^5 and F^4 respectively.

Main Scalars of a five-dimensional Finsler space and its hypersurface

The Miron frame for a five-dimensional Finsler space is constructed by the unit vectors $(l^i, m^i, n^i, p^i, q^i)$, where $l^i = y^i / L$ is the normalized supporting element, $m^i = C^i / \overline{C}$ is the normalized torsion vector ($\overline{C}$ is the length of the torsion vector C^i); n^i is constructed by $g_{ij} l^i n^j = g_{ij} m^i n^j = 0$, $g_{ij} n^i n^j = 1$, the unit vector p^i is constructed by $g_{ij} l^i p^j = g_{ij} m^i p^j = g_{ij} n^i p^j = 0$, $g_{ij} p^i p^j = 1$ and q^i is constructed by $g_{ij} l^i q^i = g_{ij} m^i q^i = g_{ij} n^i q^i = g_{ij} p^i q^i = 0, g_{ij} q^i q^j = 1$.

In the Miron frame,an arbitrary tensor $T = (T^i_j)$ is expressed in terms of scalar components as follows:

$$T^i_j = T_{\lambda\mu}\, e^i_{\lambda)}\, e_{\mu)j},$$

where $e^i_{1)} = l^i$, $e^i_{2)} = m^i$, $e^i_{3)} = n^i$, $e^i_{4)} = p^i$, $e^i_{5)} = q^i$ and the summation convection is applied to the indices λ and μ.
Let $C_{\lambda\mu\nu}$ be the scalar components of LC_{ijk} with respect to the Miron frame, i.e.

$$LC_{ijk} = C_{\lambda\mu\nu} e_{\lambda)i}\, e_{\mu)j}\, e_{\nu)k}. \tag{10}$$

M. Matsumoto [4] showed that

1. $C_{\lambda\mu\nu}$ are completely symmetric,
2. $C_{1\mu\nu} = 0$,
3. $C_{2\mu\mu} = L\tilde{C}$, $C_{3\mu\mu} = C_{4\mu\mu} = \ldots = C_{n\mu\mu} = 0$ for $n \geq 3$.

Therefore in five-dimensional Finsler space, we have

$$\left.\begin{aligned} &C_{222} + C_{233} + C_{244} + C_{255} = L\tilde{C}, \\ &C_{322} + C_{333} + C_{344} + C_{355} = 0, \\ &C_{422} + C_{433} + C_{444} + C_{455} = 0, \\ &C_{522} + C_{533} + C_{544} + C_{555} = 0. \end{aligned}\right\} \tag{11}$$

Thus putting

$$\begin{array}{lllll} C_{222} = H, & C_{233} = I, & C_{244} = K, & C_{333} = J, & C_{344} = J', \\ C_{444} = H', & C_{334} = I', & C_{234} = K', & C_{255} = M, & C_{355} = J'', \\ C_{455} = M', & C_{555} = H'', & C_{335} = I'', & C_{445} = K'', & C_{235} = N, \\ C_{245} = N', & C_{345} = M'', & & & \end{array} \tag{12$'$}$$

then, we have

$$\begin{array}{ll} H + I + K + M = LC, & C_{223} = -(J + J' + J''), \\ C_{224} = -(H' + I' + M'), & C_{225} = -(H'' + I'' + K''). \end{array}$$

Seventeen scalars $H, I, K, J, J', H', I', K', M, J'', M', H'', I'', K'', N, N', M''$ are called the main scalars of a five-dimensional Finsler space.

The equation (10) may be written in expanded form as:

$$\begin{aligned} LC_{ijk} = {} & Hm_i m_j m_k - (J + J' + J'')\Pi_{(ijk)}(m_i m_j n_k) + I\Pi_{(ijk)}(m_i n_j n_k) + J(n_i n_j n_k) \\ & - (H' + I' + M')\Pi_{(ijk)}(m_i m_j p_k) + H'(p_i p_j p_k) + K\Pi_{(ijk)}(m_i p_j p_k) \\ & - (H'' + I'' + K'')\Pi_{(ijk)}(m_i m_j q_k) + M\Pi_{(ijk)}(m_i q_j q_k) + H''(q_i q_j q_k) \\ & + I'\Pi_{(ijk)}(n_i n_j p_k) + J'\Pi_{(ijk)}(n_i p_j p_k) + I''\Pi_{(ijk)}(n_i n_j q_k) + J''\Pi_{(ijk)}(n_i q_j q_k) \\ & + K''\Pi_{(ijk)}(p_i p_j q_k) + M'\Pi_{(ijk)}(p_i q_j q_k) + K'\Pi_{(ijk)}\{m_i(n_j p_k + n_k p_j)\} \\ & + N\Pi_{(ijk)}\{m_i(n_j q_k + n_k q_j)\} + N'\Pi_{(ijk)}\{m_i(p_j q_k + p_k q_j)\} \\ & + M''\Pi_{(ijk)}\{n_i(p_j q_k + p_k q_j)\}. \end{aligned} \tag{13}$$

The hypersurface F^4 of F^5 is a four-dimensional Finsler space. The Moor frame for F^4 is given by $(l^\alpha, m^\alpha, n^\alpha, p^\alpha)$, where $l^\alpha = v^\alpha / \underline{L}$; $m^\alpha = C^\alpha / \tilde{\underline{C}}$ ($\tilde{\underline{C}}$ being the length of the torsion vector C^α of F^4), n^α is constructed by $g_{\alpha\beta} l^\alpha n^\beta = g_{\alpha\beta} m^\alpha n^\beta = 0$, $g_{\alpha\beta} n^\alpha n^\beta = 1$ and p^α is constructed by $g_{\alpha\beta} l^\alpha p^\beta = g_{\alpha\beta} m^\alpha p^\beta = g_{\alpha\beta} n^\alpha p^\beta = 0$, $g_{\alpha\beta} p^\alpha p^\beta = 1$.

For this frame, the Cartan tensor $C_{\alpha\beta\gamma}$ of F^4 is represented by [4]:

$$\begin{aligned}\underline{L}C_{\alpha\beta\gamma} &= \underline{H}m_\alpha m_\beta m_\gamma + \underline{I}\,\Pi_{(\alpha\beta\gamma)}(m_\alpha n_\beta n_\gamma) + \underline{K}\Pi_{(\alpha\beta\gamma)}(m_\alpha p_\beta p_\gamma) - (\underline{J} \\ &+ \underline{J}')\Pi_{(\alpha\beta\gamma)}(n_\alpha m_\beta m_\gamma) + \underline{J}(n_\alpha n_\beta n_\gamma) + \underline{J}'\Pi_{(\alpha\beta\gamma)}(n_\alpha p_\beta p_\gamma) - (\underline{H}' \\ &+ \underline{I}')\Pi_{(\alpha\beta\gamma)}(m_\alpha m_\beta p_\gamma) + \underline{I}'\,\Pi_{(\alpha\beta\gamma)}(n_\alpha n_\beta p_\gamma) \\ &+ \underline{H}'(p_\alpha p_\beta p_\gamma) + \underline{K}'\Pi_{(\alpha\beta\gamma)}\{m_\alpha(n_\beta p_\gamma + n_\gamma p_\beta)\}\end{aligned} \tag{14}$$

where $\Pi_{(\alpha\beta\gamma)}$ denote the cyclic interchange of α, β, γ, and summation and $\underline{H}$, $\underline{I}$, $\underline{J}$, $\underline{K}$, $\underline{J}'$, $\underline{H}'$, $\underline{I}'$ and $\underline{K}'$ are the main scalars of F^4. Transvecting (7) by v^α and using (1), we get $M_{\alpha\beta}v^\alpha = 0$, $M_\alpha v^\alpha = 0$. Therefore, $M_{\alpha\beta}$ and M_α have no component in the direction of v^α (i.e. in the direction of l^α). Also, $M_{\alpha\beta}$ is symmetric. Therefore M_α and $M_{\alpha\beta}$ may be written in the form $M_\alpha = \underline{U}m_\alpha + \underline{V}n_\alpha + \underline{W}p_\alpha$ and

$$M_{\alpha\beta} = \underline{X}m_\alpha m_\beta + \underline{Z}n_\alpha n_\beta + \underline{T}p_\alpha p_\beta + \underline{Y}(m_\alpha n_\beta + n_\alpha m_\beta + n_\alpha p_\beta + p_\alpha n_\beta + p_\alpha m_\beta + m_\alpha p_\beta).$$

Thus, we have the following:

Proposition: Let F^4 be the hypersurface of a five-dimensional Finsler space F^5, then the tensor M_α and $M_{\alpha\beta}$ defined by (7), are written as $M_\alpha = \underline{U}m_\alpha + \underline{V}n_\alpha + \underline{W}p_\alpha$ and
$M_{\alpha\beta} = \underline{X}m_\alpha m_\beta + \underline{Z}n_\alpha n_\beta + \underline{T}p_\alpha p_\beta + \underline{Y}(m_\alpha n_\beta + n_\alpha m_\beta + n_\alpha p_\beta + p_\alpha n_\beta + p_\alpha m_\beta + m_\alpha p_\beta)$ respectively.

From (13) and (14), the torsion vector C_i and C_α are represented by $LC_i = (H + I + K + M)\,m_i$ and $\underline{L}C_\alpha = (\underline{H} + \underline{I} + \underline{K})\,m_\alpha$ respectively. The equation (19) and proposition lead to

$$\begin{aligned}m_i &= (H + I + K + M)^{-1}\{(\underline{H} + \underline{I} + \underline{K} + \underline{U})\,m_\alpha B_i^\alpha + \underline{V}n_\alpha B_i^\alpha + \underline{W}p_\alpha B_i^\alpha \\ &+ (M + \underline{X} + \underline{Z} + \underline{T})\,B_i\},\end{aligned} \tag{15}$$

which yields

$$(H + I + K + M)^2 = (\underline{H} + \underline{I} + \underline{U} + \underline{K})^2 + \underline{V}^2 + \underline{W}^2 + (M + \underline{X} + \underline{Z} + \underline{T})^2. \tag{16}$$

Let us put

$$\begin{aligned}&(H + I + K + M)^{-1}(\underline{H} + \underline{I} + \underline{U} + \underline{K}) = a, \\ &(H + I + K + M)^{-1}\underline{V} = b, \\ &(H + I + K + M)^{-1}\underline{W} = d, \\ &(H + I + K + M)^{-1}(M + \underline{X} + \underline{Z} + \underline{T}) = t,\end{aligned}$$

then, $m_i = am_\alpha B_i^\alpha + bn_\alpha B_i^\alpha + dp_\alpha B_i^\alpha + tB_i.$ (17)

Let us write the unit vectors n_i, p_i and q_i as:

$$n_i = em_\alpha B_i^\alpha + fn_\alpha B_i^\alpha + gp_\alpha B_i^\alpha + hB_i, \tag{18}$$

$$p_i = a'm_\alpha B_i^\alpha + b'n_\alpha B_i^\alpha + d'p_\alpha B_i^\alpha + t'B_i, \tag{19}$$

And $q_i = e'm_\alpha B_i^\alpha + f'n_\alpha B_i^\alpha + g'p_\alpha B_i^\alpha + h'B_i,$ (20)

where $a, b, d, t, e, f, g, h, a', b', d', t', e', f', g', h'$ are given by

$$\left.\begin{aligned}&ae + bf + dg + th = 0 \\ &aa' + bb' + dd' + tt' = 0 \\ &ae' + bf' + dg' + th' = 0 \\ &ea' + fb' + gd' + ht' = 0 \\ &ee' + ff' + gg' + hh' = 0 \\ &a'e' + b'f' + d'g' + t'h' = 0\end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned}&a^2 + b^2 + d^2 + t^2 = 1, \\ &e^2 + f^2 + g^2 + h^2 = 1, \\ &a'^2 + b'^2 + d'^2 + t'^2 = 1, \\ &e'^2 + f'^2 + g'^2 + h'^2 = 1.\end{aligned}\right\} \tag{21}$$

From (21), we also have the relations:

$$\left.\begin{aligned} ab+ef+a'b'+e'f' &= 0\\ ad+eg+a'd'+e'g' &= 0\\ at+eh+a't'+e'h' &= 0\\ bd+fg+b'd'+f'g' &= 0\\ dt+gh+d't'+g'h' &= 0\end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} a^2+e^2+a'^2+e'^2 &= 1,\\ b^2+f^2+b'^2+f'^2 &= 1,\\ d^2+g^2+d'^2+g'^2 &= 1,\\ t^2+h^2+t'^2+h'^2 &= 1.\end{aligned}\right\} \tag{21'}$$

Substituting (17), (18), (19) into (13) and using (3), we get

$$\begin{aligned}\underline{L}C_{\alpha\beta\gamma} = {} & m_\alpha m_\beta m_\gamma \{a^3H + e(e^2-3a^2)J - 3e(a^2-a'^2-a'b')J' - 3(a^2e-e'^3)J''\\
& + a'(a'^2-3a^2)H' + e'(e'^2-3a^2)H'' + 3ae^2I + 3a'(e^2-a^2)I' + 3e'(e^2\\
& - a^2)I'' + 3aa'^2K + 6aa'eK' + 3e'(a'^2-a^2)K'' + 3ae'^2M + 3a'(e'^2\\
& - a^2)M' + 6a'ee'M'' + 6aee'N + 6aa'e'N'\} + (n_\alpha n_\beta m_\gamma + n_\alpha n_\gamma m_\beta + n_\beta\\
& n_\gamma m_\alpha)\{ab^2H + (a'b'^2 - 2abb' - a'b^2)H' - (b^2e' + 2abf' - e'f'^2)H''\\
& + f(af+2be)I + (a'f^2 + 2efb' - a'b^2 - 2abb')I' + (e'f^2 + 2ff'e\\
& - b^2e' - 2abf')I'' + (ef^2 - b^2e - 2abf)J + (a'b'f + eb'^2 - b^2e\\
& - 2abf)J' + (ef'^2 + 2ff'e' - b^2e - 2abf)J'' + b'(ab' + 2a'b)K\\
& + 2(afn' + a'b'f + b'^2e)K' + (b'^2e' + 2a'b'f' - b^2e' - 2abf')K''\\
& + (af'^2 + 2be'f')M + (a'f'^2 + 2b'f'e' - a'b^2 - 2abb')M' + 2(b'ef'\\
& + b'e'f + a'ff')M'' + 2(bef' + bfe' + aff')N + 2(ab'f' + bb'e'\\
& + a'bf')N'\} + (n_\alpha m_\beta m_\gamma + m_\alpha n_\beta m_\gamma + m_\alpha m_\beta n_\gamma)\{a^2bH + (a'^2b' - a^2b'\\
& - 2aa'b)H' + (e'^2f' - a^2f' - 2abe' - 2aba')H'' + e(be+2af)I\\
& + (b'e^2 + 2a'ef - a^2b' - 2aa'b)I' + (e^2f' + 2efe' - a^2f' - 2abe')I''\\
& + (e^2f - a^2f - 2abe)J + (a'b'f + a'b'e + a'^2f - a^2f - 2abe)J'\\
& + (e'^2f + 2ee'f' - a^2f - 2abe)J'' + a'(a'b + 2ab')K + 2(aa'f\\
& + ab'e + a'b'e - aba')K' + (2a'b'e' + a'^2f' - a^2f'^2 - 2abe')K''\\
& + (be'^2 + 2af'e')M + (b'e'^2 + 2a'f'e' - a^2b')M' + 2(b'e'e + a'ef'\\
& + a'e'f)M'' + 2(aef' + bee' + ae'f)N + 2(ab'e' + aa'f' + a'be')N'\}\\
& + (p_\alpha m_\beta m_\gamma + m_\alpha p_\beta m_\gamma + m_\alpha m_\beta p_\gamma)\{a^2dH + (a'^2d' - a^2d' - 2aa'd)H'\\
& + (e'^2g' - a^2g' - 2ade')H'' + e(de+2ag)I + (d'e^2 + 2a'eg - 2a^2d'\\
& - 2aa'd)I' + (e^2g' + 2ee'g - a^2g' - 2ade')I'' + (e^2g - a^2g - 2ade)J\end{aligned} \tag{22}$$

$$+(a'd'e+a'b'g-a^2g-2ade)J'+(e'^2g+2ee'g'-a^2g-2ade)J''+a'(a'd$$
$$+2ad')K+2(aa'g+ad'e+a'de-aa'd)K'+(2a'd'e'+a'^2g'-a^2g'$$
$$-2ade')K''+e'(de'+2ag')M+e'(d'e'+2a'g')M'+2(d'e'e+a'eg'$$
$$+ae'g)M''+2(aeg'+ae'g+dee')N+2(aa'g'+a'de'+ad'e')N'\}$$
$$+(p_\alpha p_\beta m_\gamma+p_\alpha m_\beta p_\gamma+m_\alpha p_\beta p_\gamma)\{ad^2H+(a'd'^2-a'd^2-2aad')H'+(e'g'^2$$
$$-d^2e'-2adg')H''+(ag^2+2deg)I+(a'g^2+2ed'g-2add'-a'd^2)I'$$
$$+(e'g^2+2egg'-d^2e'-2adg')I''+(eg^2-d^2g-2adg)J+(d'^2e+2a'd'g$$
$$-ed^2-2adg)J'+(eg'^2+2e'gg'-ed^2-2adg)J''+d'(ad'+2a'd)K$$
$$+2(ad'e+a'dg+ad'g)K'+(d'^2e'+2a'd'g-d^2e'-2adg')K''+g'(ag'$$
$$+2de')M+g'(a'g'+2d'e')M'+2(d'eg'+d'e'g+a'gg')M''+2(agg'$$
$$+deg'+de'g)N+2(ad'g'+dd'e'+a'dg')N'\}+(p_\alpha n_\beta n_\gamma+n_\alpha p_\beta n_\gamma+n_\alpha n_\beta$$
$$p_\gamma)\{b^2dH+(b'^2d'-2bb'd-b^2d')H'+(g'f'^2-b^2g'-2bdf')H''+f(df$$
$$+2bg)I+(d'f^2+2fgb'-2bb'd-d'b^2)I'+(f^2g'+2ff'g-b^2g'-2b$$
$$df')I''+(gf^2-b^2g-2bdf)J+(gb'^2+2fb'd'-b^2g-2bdf)J'+(gf'^2$$
$$+2g'f'f-b^2g-2bdf)J''+b'(b'd+2bd')K+2(b'^2g+b'fd'+dfn')K'$$
$$+(b'^2g'+2b'd'f'-b^2g'-2bdf')K''+f'(df'+2bg')M+(d'f'g'$$
$$+2bg'f'-2bb'd-b^2d')M'+2(d'f'f+b'g'f+b'f'g)M''+2(bfg'$$
$$+bgf'+dff')N+2(d'bf'+bb'g'+b'df')N'\}+(p_\alpha p_\beta n_\gamma+p_\alpha n_\beta p_\gamma$$
$$+n_\alpha p_\beta p_\gamma)\{bd^2H+(b'd'^2-2bdd'-d^2b')H'+(f'g'^2-d^2f'-2bdg')H''$$
$$+g(bg+2df)I+(b'g^2+2fgd'-2bdd'-b'd^2)I'+(g^2f'+2fgg''-d^2f'$$
$$-2bdg')I''+(fg^2-d^2f-2bdg)J+(fd'^2+2b'gd'-d^2f-2bdg)J'+$$
$$(fg'^2+2gg'f'-d^2f-2bdg)J''+d'(bd'+2b'd)K+2(dfd'+b'dg+$$
$$b'd'g)K'+(f'd'^2+2b'd'g')K''+g'(bg'+2df')M+(b'g'^2+2d'g'f'$$
$$-2bdd'-b'd^2)M'+2(d'fg'+d'gf'+b'gg')M''+2(bgg'+dfg'+$$
$$dgf')N+2(bd'g'+dd'f'+b'dg')N'\}+n_\alpha n_\beta n_\gamma\{b^3H+b'(b'^2-3b^2)H'$$
$$+f'(f'^2-3b^2)H''+3bf^2I+3b'(f^2-b^2)I'+3f'(f^2-b^2)I''+f(f^2$$
$$-3b^2)J+3f(b'^2-b^2)J'+3f(f'^2-b^2)J''+3bb'^2K+6b'fn'K'+3f'(b'^2$$
$$-b^2)K''+3bf'^2M+3b'(f'^2-b^2)M'+6b'ff'M''+2(d'fg'+d'gf'$$
$$+b'gg')M''+2(bgg'+dfg'+dgf')N+2(bd'g'+dd'f'+b'dg')N'\}$$
$$+n_\alpha n_\beta n_\gamma\{b^3H+b'(b'^2-3b^2)H'+f'(f'^2-3b^2)H''+3bf^2I+3b'(f^2$$

$$-b^2)I' + 3f'(f^2 - b^2)I'' + f(f^2 - 3b^2)J + 3f(b'^2 - b^2)J' + 3f(f'^2 - b^2)J''$$
$$+ 3bb'^2K + 6b'fn'K' + 3f'(b'^2 - b^2)K'' + 3bf'^2M + 3b'(f'^2 - b^2)M'$$
$$+ 6b'ff'M'' + 6bff'N + 6bb'f'N'\} + p_\alpha p_\beta p_\gamma\{d^3H + d'(d'^2 - 3d^2)H'$$
$$+ g'(g'^2 - 3d^2)H'' + 3dfgI + 3d'(g^2 - d^2)I' + 3g'(g^2 - d^2)I'' + g(g^2$$
$$- 3d^2)J + 3g(d'^2 - d^2)J' + 3g(g'^2 - d^2)J'' + 3dd'^2K + 6dd'gK' + 3g'(d'^2$$
$$- d^2)K'' + 3dg'^2M + 3d'(g'^2 - d^2)M' + 6d'gg'M'' + 6dgg'N + 6dd'g'N'\}$$
$$+ \{m_\alpha(n_\beta p_\gamma + n_\gamma p_\beta) + m_\beta(n_\gamma p_\alpha + n_\alpha p_\gamma) + m_\gamma(n_\alpha p_\beta + n_\beta p_\alpha)\}[abdH - (J + J'$$
$$+ J'')(bde + adf + abg) + I(beg + edf + afg) + J(egf) - (H' + I' + M')$$
$$(abd' + ab'd + a'bd) + a'b'd'H' + K(a'b'd + a'bd' + ab'd') - (H'' + I''$$
$$+ K'')(abg' + adf' + bde') + M(ag'f' + be'g' + de'f') + e'g'f'H''$$
$$+ I'(d'ef + b'eg + a'fg) + J'(b'd'e + a'd'f + a'b'g) + I''(efg' + f'eg$$
$$+ fge') + J''(e'f'g + ef'g' + fe'g') + K''(a'b'g' + a'd'f' + b'd'e') + M'(a'$$
$$g'f' + b'e'g' + d'e'f') + K'(ab'g + ad'f + b'ed' + a'b'g + a'df + b'ed)$$
$$+ N\{afg' + af'g + af'g + afg' + beg' + be'g\} + N'\{ad'f' + ab'g' + a'g'b$$
$$+ bd'e' + a'df' + b'de'\} + M''\{ed'f' + eb'g' + a'g'f + fd'e' + a'gf'$$
$$+ b'ge'\}].$$

Comparing above equation with (14), we get

$$\begin{aligned}\underline{H} = {} & a^3H + e(e^2 - 3a^2)J - 3e(a^2 - a'^2 - a'b')J' - 3(a^2e - e'^3)J'' + a'(a'^2 \\ & - 3a^2)H' + e'(e'^2 - 3a^2)H'' + 3ae^2I + 3a'(e^2 - a^2)I' + 3e'(e^2 - a^2)I'' \\ & + 3aa'^2K + 6aa'eK' + 3e'(a'^2 - a^2)K'' + 3ae'^2M + 3a'(e'^2 - a^2)M' \\ & + 6a'ee'M'' + 6aee'N + 6aa'e'N'\end{aligned} \tag{23}$$

$$\begin{aligned}\underline{I} = {} & ab^2H + (a'b'^2 - 2abb' - a'b^2)H' - (b^2e' + 2abf' - e'f'^2)H'' + f(af \\ & + 2be)I + (a'f^2 + 2efb' - a'b^2 - 2abb')I' + (e'f^2 + 2ff'e - b^2e' \\ & - 2abf')I'' + (ef^2 - b^2e - 2abf)J + (a'b'f + eb'^2 - b^2e - 2abf)J' \\ & + (ef'^2 + 2ff'e' - b^2e - 2abf)J'' + b'(ab' + 2a'b)K + 2(afn' + a'b'f \\ & + b'^2e)K' + (b'^2e' + 2a'b'f' - b^2e' - 2abf')K'' + (af'^2 + 2be'f')M \\ & + (a'f'^2 + 2b'f'e' - a'b^2 - 2abb')M' + 2(b'ef' + b'e'f + a'ff')M'' \\ & + 2(bef' + bfe' + aff')N + 2(ab'f' + bb'e' + a'bf')N'\end{aligned} \tag{24}$$

$$\begin{aligned}\underline{K} = {} & ad^2H + (a'd'^2 - 2add' - a'd^2)H' + (e'g'^2 - 2adg' - e'd^2)H'' + (ag^2 \\ & + 2deg)I + (a'g^2 + 2ed'g - a'd^2 - 2add')I' + (e'g^2 + 2egg' - d^2e' \\ & - 2adg')I'' + (eg^2 - d^2g - 2adg)J + (d'^2e + 2a'd'g - ed^2 - 2adg)J' \\ & + (eg'^2 + 2e'gg' - ed^2 - 2adg)J'' + d'(ad' + 2a'd)K + 2(ad'e + a'dg \\ & + ad'g)K' + (d'^2e' + 2a'd'g - d^2e' - 2adg')K'' + g'(ag' + 2de')M \\ & + g'(a'g' + 2d'e')M' + 2(d'eg' + d'e'g + d'gg')M'' + 2(agg' + deg' \\ & + de'g)N + 2(ad'g' + dd'e' + a'dg')N'\end{aligned} \tag{25}$$

$$\begin{aligned}-(J+J^{'}) &= a^2bH+(a^{'2}b^{'}-a^2b^{'}-2aa^{'}b)H^{'}+(e^{'2}f^{'}-a^2f^{'}-2abe^{'}\\ &-2aba^{'})H^{''}+e(be+2af)I+(b^{'}e^2+2a^{'}ef-a^2b^{'}-2aa^{'}b)I^{'}\\ &+(e^2f^{'}+2efe^{'}-a^2f^{'}-2abe^{'})I^{''}+(e^2f-a^2f-2abe)J\\ &+(a^{'}b^{'}f+a^{'}b^{'}e+a^{'2}f-a^2f-2abe)J^{'}+(e^{'2}f+2ee^{'}f\\ &-a^2f-2abe)J^{''}+a^{'}(a^{'}b+2ab^{'})K+2(aa^{'}f+ab^{'}e\\ &+a^{'}b^{'}e-aba^{'})K^{'}+(2a^{'}b^{'}e^{'}+a^{'2}f^{'}-a^2f^{'2}-2abe^{'})K^{''}\\ &+(be^{'2}+2af^{'}e^{'})M+(b^{'}e^{'2}+2a^{'}f^{'}e^{'}-a^2b^{'})M^{'}\\ &+2(b^{'}e^{'}e+a^{'}ef^{'}+a^{'}e^{'}f)M^{''}+2(aef^{'}+bee^{'}\\ &+ae^{'}f)N+2(ab^{'}e^{'}+aa^{'}f^{'}+a^{'}be^{'})N\end{aligned} \tag{26}$$

$$\begin{aligned}\underline{J} &= b^3H+b^{'}(b^{'2}-3b^2)H^{'}+f^{'}(f^{'2}-3b^2)H^{''}+3bf^2I+3b^{'}(f^2-b^2)I^{'}\\ &+3f^{'}(f^2-b^2)I^{''}+f(f^2-3b^2)J+3f(b^{'2}-b^2)J^{'}+3f(f^{'2}-b^2)J^{''}\\ &+3bb^{'2}K+6b^{'}fn^{'}K^{'}+3f^{'}(b^{'2}-b^2)K^{''}+3bf^{'2}M+3b^{'}(f^{'2}-b^2)M^{'}\\ &+6b^{'}ff^{'}M^{''}+6bff^{'}N+6bb^{'}f^{'}N^{'}\end{aligned} \tag{27}$$

$$\begin{aligned}\underline{J}^{'} &= bd^2H+(b^{'}d^{'2}-2bdd^{'}-b^{'}d^2)H^{'}+(f^{'}g^{'2}-2bdg^{'}-f^{'}d^2)H^{''}+g(bg\\ &+2df)I+(b^{'}g^2+2fgd^{'}-b^{'}d^2-2bdd^{'})I^{'}+(f^{'}g^2+2fgg^{''}-d^2f^{'}\\ &-2bdg^{'})I^{''}+(fg^2-d^2f-2bdg)J+(d^{'2}f+2b^{'}d^{'}g-fd^2-2bdg)J^{'}\\ &+(fg^{'2}+2f^{'}gg^{'}-fd^2-2bdg)J^{''}+d^{'}(bd^{'}+2b^{'}d)K+2(dfd^{'}+b^{'}dg\\ &+b^{'}d^{'}g)K^{'}+(d^{'2}f^{'}+2b^{'}d^{'}g^{'})K^{''}+g^{'}(bg^{'}+2df^{'})M+(b^{'}g^{'2}\\ &+2d^{'}g^{'}f^{'}-b^{'}d^2-2bdd^{'})M^{'}+2(d^{'}fg^{'}+d^{'}gf^{'}+b^{'}gg^{'})M^{''}\\ &+2(bgg^{'}+dfg^{'}+dgf^{'})N+2(bd^{'}g^{'}+dd^{'}f^{'}+b^{'}dg^{'})N^{'}\end{aligned} \tag{28}$$

$$\begin{aligned}-(\underline{H}^{'}+\underline{I}^{'}) &= a^2dH+(a^{'2}d^{'}-a^2d^{'}-2aa^{'}d)H^{'}+(e^{'2}g^{'}-a^2g^{'}-2ade^{'})H^{''}\\ &+e(de+2ag)I+(d^{'}e^2+2a^{'}eg-2a^2d^{'}-2aa^{'}d)I^{'}+(e^2g^{'}\\ &+2ee^{'}g-a^2g^{'}-2ade^{'})I^{''}+(e^2g-a^2g-2ade)J+(2a^{'}d^{'}e\\ &+a^{'}b^{'}g-a^2g-2ade)J^{'}+(e^{'2}g+2ee^{'}g^{'}-a^2g-2ade)J^{''}\\ &+a^{'}(a^{'}d+2ad^{'})K+2(aa^{'}g+ad^{'}e+a^{'}de-aa^{'}d)K^{'}\\ &+(a^{'2}g^{'}+2a^{'}d^{'}e^{'}-a^2g^{'}-2ade^{'})K^{''}+e^{'}(de^{'}+2ag^{'})M\\ &+e^{'}(d^{'}e^{'}+2a^{'}g^{'})M^{'}+2(d^{'}ee^{'}+a^{'}eg^{'}+ae^{'}g)M^{''}+2(aeg^{'}\\ &+ae^{'}g+dee^{'})N+2(aa^{'}g^{'}+a^{'}de^{'}+ad^{'}e^{'})N^{'}\end{aligned} \tag{29}$$

$$\begin{aligned}\underline{I}^{'} &= b^2dH+(b^{'2}d^{'}-2bb^{'}d-b^2d^{'})H^{'}+(g^{'}f^{'2}-2bdf^{'}-b^2g^{'})H^{''}+f(df\\ &+2bg)I+(d^{'}f^2+2fgb^{'}-d^{'}b^2-2bb^{'}d)I^{'}+(f^2g^{'}+2ff^{'}g-b^2g^{'}\\ &-2bdf^{'})I^{''}+(gf^2-b^2g-2bdf)J+(gb^{'2}+2fb^{'}d^{'}-b^2g-2bdf)J^{'}\\ &+(2g^{'}f^{'}f+gf^{'2}-b^2g-2bdf)J^{''}+b^{'}(b^{'}d+2bd^{'})K+2(b^{'2}g\\ &+b^{'}fd^{'}+dfn^{'})K^{'}+(b^{'2}g^{'}+2b^{'}d^{'}f^{'}-b^2g^{'}-2bdf^{'})K^{''}+f^{'}(df^{'}\\ &+2bg^{'})M+(d^{'}f^{'}g^{'}+2bg^{'}f^{'}-2bb^{'}d-b^2d^{'})M^{'}+2(d^{'}f^{'}f\\ &+b^{'}g^{'}f+b^{'}f^{'}g)M^{''}+2(bfg^{'}+bgf^{'}+dff^{'})N+2(d^{'}bf^{'}\\ &+bb^{'}g^{'}+b^{'}df^{'})N^{'}\end{aligned} \tag{30}$$

$$\begin{aligned}\underline{H}' =\ & d^3H + d'(d'^2-3d^2)H' + g'(g'^2-3d^2)H'' + 3dfgI + 3d'(g^2-d^2)I' \\ & + 3g'(g^2-d^2)I'' + g(g^2-3d^2)J + 3g(d'^2-d^2)J' + 3g(g'^2-d^2)J'' \\ & + 3dd'^2K + 6dd'gK' + 3g'(d'^2-d^2)K'' + 3dg'^2M + 3d'(g'^2-d^2)M' \\ & + 6d'ggM'' + 6dgg'N + 6dd'g'N' \end{aligned} \tag{31}$$

$$\begin{aligned}\underline{K}' =\ & abdH + (a'b'd - abd' - ab'd - a'bd)H' + (e'g'f' - abg' - adf' \\ & - bde')H'' + (beg + edf + afg)I + (d'ef + b'eg + a'fg - abd' - ab'd \\ & - a'bd)I' + (efg' + ef'g + fge' - abg' - adf' - bde')I'' + (egf - bde \\ & - adf - abg)J + (b'd'e + a'd'f + a'b'g - bde - adf - abg)J' + (e'f'g \\ & + ef'g' + fe'g' - bde - adf - abg)J'' + (a'b'd + a'bd' + ab'd')K \\ & + (ab'g + ad'f + b'ed' + a'b'g + a'df + b'ed)K' + (a'b'g' + a'd'f' \\ & + b'd'e' - abg' - adf' - bde')K'' + (ag'f' + be'g' + de'f')M \\ & + (a'g'f' + b'e'g' + d'e'f' - abd' - ab'd - a'bd)M' + (ed'f' \\ & + eb'g' + a'g'f + fd'e' + a'gf' + b'ge')M'' + 2(afg' + af'g \\ & + beg')N + (ad'f' + ab'g' + ab'g' + a'g'b + bd'e' + a'df' \\ & + b'de')N' \end{aligned} \tag{32}$$

Similarly, substituting (17), (18), (19) into (13) and using (7) and proposition, we get

$$\begin{aligned}\underline{X} =\ & a^2tH + (a'^2t' - a^2t' - 2aa't)H' + (e'^2h' - a^2h' - 2ae't)H'' + (e^2t + 2aeh)I \\ & + (e^2t' + 2a'eh - a^2t' - 2aa't)I' + (e^2h' + 2ee'h - a^2h' - 2ae't)I'' + (e^2h \\ & - a^2h - 2aet)J + (2a'et' + a'^2h - a^2h - 2aet)J' + (2ee'h' + e'^2h - a^2h \\ & - 2aet)J'' + (2aa't' + a'^2t)K + 2(aet' + a'et + aa'h)K' + 2(a'^2h' + 2a'e't' \\ & - a^2h' - 2ae't)K'' + (e'^2t + 2ae'h')M + (e'^2t' + 2a'e'h' - a^2t' - 2aa't)M' \\ & + 2(a'e't + a'eh' + ee't')M'' + 2(ae'h + aeh' + ee't)N + 2(aa'h' + ae't' \\ & + a'e't)N' \end{aligned} \tag{33}$$

$$\begin{aligned}\underline{Z} =\ & b^2tH + (b'^2t' - b^2t' - 2bb't)H' + (f'^2h' - b^2h' - 2bf't)H'' + (f^2t + 2bfh)I \\ & + (f^2t' + 2b'fh - b^2t' - 2bb't)I' + (f^2h' + 2ff'h - b^2h' - 2bf't)I'' + (f^2h \\ & - b^2h - 2bft)J + (b'^2h + 2b't'f - b^2h - 2bft)J' + (f'^2h + 2ff'h' - b^2h \\ & - 2bft)J'' + (b'^2t + 2bb't')K + 2(bft' + bb'h + b'ft)K' + (b'^2h' + 2b'f't' \\ & - b^2h' - 2bf't)K'' + (f'^2t + 2bf'h')M + (f'^2t' + 2b'f'h' - b^2t' \\ & - 2bb't)M' + 2(ff't' + b'fh' + b'f'h)M'' + 2(bfh' + bhf' \\ & + ff't)N + 2(bb'h' + bf't' + f'b't)N' \end{aligned} \tag{34}$$

$$\begin{aligned}\underline{T} =\ & d^2tH + (d'^2t' - d^2t' - 2dd't)H' + (g'^2h' - d^2h' - 2dg't)H'' + (g^2t \\ & + 2dgh)I + (g^2t' + 2d'gh - d^2t' - 2dd't)I' + (g^2h' + 2gg'h - d^2h' \\ & - 2dg't)I'' + (g^2h - d^2h - 2dgt)J + (d'^2h + 2d'gt' - d^2h - 2dgt)J' \\ & + (g'^2h + 2gg'h' - d^2h - 2dgt)J'' + (d'^2t + 2dd't')K + 2(dgt' + dd'h \\ & + d'gt)K' + (d'^2h' + 2d'g't' - d^2h' - 2dg't)K'' + (g'^2t + 2dg'h')M \\ & + (g'^2t' + 2d'g'h' - d^2t' - 2dd't)M' + 2(d'gh' + d'g'h + gg't')M'' \\ & + 2(dgh' + dhg' + gg't)N + 2(dd'h' + d'g't + dg't')N' \end{aligned} \tag{35}$$

$$\begin{aligned}\underline{Y} &= abtH + (a'b't' - abt' - ab't - a'bt)H' + (e'f'h' - abh' - af't - be't)H'' \\ &+ (afh + beh + eft)I + (eft' + b'eh + a'fh - abt' - ab't - a'bt)I' + (feh' \\ &+ ef'h + e'fh - abh' - af't - be't)I'' + (efh - abh - aft - bet)J + (a'ft' \\ &+ a'b'h + b'et' - abh - aft - bet)J' + (ef'h' + e'fh' + e'f'h - abh - aft \\ &- bet)J'' + (ab't' + a'bt' + a'b't)K + (ab'h + a'bh + b'et + a'ft + aft' \\ &+ bet')K' + (a'b'h' + a'f't' + b'e't' - abh' - af't - be't)K'' + (af'h' \\ &+ be'h' + e'f't)M + (a'f'h' + b'e'h' + e'f't' - abt' - ab't - a'bt)M' \\ &+ (ef't' + a'f'h + a'fh' + b'eh' + b'e'h)M'' + (afh' + af'h + beh' \\ &+ be'h + ef't + e'ft)N + (ef't' + a'f'h + a'fh' + b'eh' + e'ft' + b'e'h)N'\end{aligned} \tag{36}$$

$$\begin{aligned}\underline{S} &= adtH + (a'd't' - bdt' - a'dt - ad't)H' + (e'g'h' - ag't - de't - adh')H'' \\ &+ (egt + deh + agh)I + (egt' + a'gh + ee'h - bdt' - a'dt - ad't)I' + (geh' \\ &+ eg'h + e'gh - ag't - de't - adh')I'' + (egh - adh - agt - det)J + (a'gt' \\ &+ d'et' + a'd'h - adh - agt - det)J' + (e'gh' + eg'h' + e'g'h - adh - agt \\ &- det)J'' + (a'd't + ad't + a'dt')K + (a'dh + ad'h + agt' + a'gt + det' \\ &+ d'et)K' + (a'd'h' + a'g't' + d'e't' - ag't - de't - adh')K'' + (ag'h' \\ &+ de'h' + e'g't)M + (a'g'h' + d'e'h' + e'g't' - bdt' - a'dt - ad't)M' \\ &+ (d'eh' + a'gh' + eg't' + e'gt' + a'g'h + d'e'h)M'' + (deh' + agh' \\ &+ ag'h + de'h + eg't + e'gt)N + (ad'h' + a'dh' + ag't' + de't' \\ &+ a'g't + d'e't)N'\end{aligned} \tag{37}$$

$$\begin{aligned}\underline{U} &= at^2H + (a't'^2 - a't^2 - 2att')H' + (e'h'^2 - e't^2 - 2ath')H'' + (ah^2 + 2eth)I \\ &+ (a'h^2 + 2eth' - a't^2 - 2att')I' + (e'h^2 + 2ehh' - e't^2 - 2ath')I'' + (eh^2 \\ &- et^2 - 2ath)J + (et'^2 + 2a'ht' - et^2 - 2ath)J' + (eh'^2 + 2e'hh' - et^2 \\ &- 2ath)J'' + (at'^2 + 2a'tt')K + 2(aht' + a'th + ett')K' + (e't'^2 + 2a't'h' \\ &- e't^2 - 2ath')K'' + (ah'^2 + 2e'th')M + (2t'h'^2 + a'hh' - a't^2 - 2att')M' \\ &+ 2(et'h' + e't'h + a'hh')M'' + 2(ahh' + e'th + eth')N + 2(at'h' + e'tt' \\ &+ a'h't)N'\end{aligned} \tag{38}$$

$$\begin{aligned}\underline{V} &= bt^2H + (b't'^2 - b't^2 - 2btt')H' + (f'h'^2 - f't^2 - 2bth')H'' + (bh^2 \\ &+ 2fth)I + (b'h^2 + 2fth' - b't^2 - 2btt')I' + (f'h^2 + 2fhh' - f't^2 \\ &- 2bth')I'' + (fh^2 - ft^2 - 2bth)J + (ft'^2 + 2b'ht' - ft^2 - 2bth)J' \\ &+ (fh'^2 + 2f'hh' - ft^2 - 2bth)J'' + (bt'^2 + 2b'tt')K + 2(bht' + b'th \\ &+ ftt')K' + (f't'^2 + 2b't'h' - f't^2 - 2bth')K'' + (bh'^2 + 2f'th')M \\ &+ (2t'h'^2 + b'hh' - b't^2 - 2btt')M' + 2(ft'h' + f't'h + b'hh')M'' \\ &+ 2(bhh' + f'th + fth')N + 2(bt'h' + f'tt' + b'h't)N'\end{aligned} \tag{39}$$

$$\begin{aligned}\underline{W} &= dt^2H + (d't'^2 - d't^2 - 2dtt')H' + (g'h'^2 - g't^2 - 2dth')H'' + (dh^2 \\ &+ 2gth)I + (d'h^2 + 2gth' - d't^2 - 2dtt')I' + (g'h^2 + 2ghh' - g't^2\end{aligned} \tag{40}$$

$$-2dth')I'' + (gh^2 - gt^2 - 2dth)J + (gt'^2 + 2d'ht' - gt^2 - 2dth)J'$$
$$+ (gh'^2 + 2g'hh' - gt^2 - 2dth)J'' + (dt'^2 + 2d'tt')K + 2(dht' + d'th$$
$$+ gtt')K' + (g't'^2 + 2d't'h' - g't^2 - 2dth')K'' + (dh'^2 + 2g'th')M$$
$$+ (2t'h'^2 + d'hh' - d't^2 - 2dtt')M' + 2(gt'h' + g't'h + d'hh')M''$$
$$+ 2(dhh' + g'th + gth')N + 2(dt'h' + g'tt' + g'h't)N'$$

$$\begin{aligned} M =\,& t^3H + 3(t'^3 - t^2t')H' + (h'^3 - 3h't^2)H'' + 3th^2I + 3(h^2t' - t^2t')I' + 3(h^2h' \\ & - h't^2)I'' + (h^3 - 3t^2h)J + 3(ht'^2 - t^2h)J' + 3(hh'^2 - t^2h)J'' + 3tt'^2K \\ & + 6tt'hK' + 3(h't'^2 - h't^2)K'' + 3th'^2M + 3t'h'^2M' + 6ht'h'M'' \\ & + 6thh'N + 6tt'h'N'. \end{aligned} \tag{41}$$

Thus, we have

Theorem: Let F^4 be the hypersurface of a four-dimensional Finsler space F^4, then the main scalars of F^4 and F^5 are related by (23), (24), (25), (26), (27), (28), (29), (30), (31), (32).

C-Reducible Finsler Space

A Finsler space of dimension $n(n > 2)$ is called C-reducible if C_{ijk} is written as

$$C_{ijk} = (C_ih_{jk} + C_jh_{ki} + C_kh_{ij})/(n+1),$$

where $h_{ij}(= g_{ij} - l_il_j)$ is the angular metric tensor. Since $\delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta}$ are scalar components of h_{ij} with respect to the Miron's frame $\{e^i_{(\alpha)}\}$ of F^5, therefore for a five-dimensional C-reducible Finsler space, we have

$$C_{\alpha\beta\gamma} = LC\{\delta_{2\alpha}(\delta_{\beta\gamma} - \delta_{1\beta}\delta_{1\gamma}) + \delta_{2\beta}(\delta_{\gamma\alpha} - \delta_{1\gamma}\delta_{1\alpha}) + \delta_{2\gamma}(\delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta})\}/6. \tag{42}$$

In view of notations given in equation (12)′ above equation gives

$$\begin{aligned} H = 3I = 3K = 3M, \qquad & H' = H'' = I' = I'' = K' = K'' = M' \\ & = M'' = N = N' = J = J' = J'' = 0. \end{aligned} \tag{43}$$

In a five-dimensional Finsler space, $H + I + K + M = LC$ is called unified main scalars. If the unified main scalar is constant, i.e. *LC* is constant, then *H*, *I*, *K*, *M* are constant and we have the following theorem:

Theorem: In a five-dimensional C-reducible Finsler space with non-zero constant unified main scalar, the main scalars *H*, *I*, *K*, *M* are constants and all the remaining main scalars vanish.

In view of (21), (22) and (43), equations (23) to (41) reduce to

$$\begin{aligned} & \underline{H} = aH, \quad \underline{I} = aH/3, \quad \underline{K} = aH/3, \quad \underline{X} = tH/3, \quad \underline{Z} = tH/3, \\ & \underline{J} = \underline{J}' = \underline{H}' = \underline{I}' = \underline{K}' = 0, \quad \underline{T} = tH/3, \quad \underline{Y} = 0, \quad \underline{S} = 0, \\ & \underline{U} = aH/3, \quad \underline{V} = 0, \quad \underline{W} = 0, \quad M = tH, \end{aligned} \tag{44}$$

which gives $\underline{H} = 3\underline{I} = 3\underline{K}$, $\underline{J} = \underline{J}' = \underline{H}' = \underline{I}' = \underline{K}' = 0$. This shows that the hypersurface F^4 of a five-dimensional C-reducible Finsler space F^5 is also C-reducible, which is in agreement to the Matsumoto's results [3].

In view of (43) and (44), equation (16) becomes

$$(2H)^2 = (2\underline{H})^2 + 0 + 0 + (2M)^2,$$

which gives

$$\underline{H} = \pm\sqrt{H^2 - M^2}.$$

Consequently, we have

Theorem: Let F^4 be the hypersurface of a five-dimensional C-reducible Finsler space F^5, then for the function *M* defined by (7), the main scalars $\underline{H}$, $\underline{I}$, $\underline{K}$, $\underline{J}$, $\underline{J}'$, $\underline{H}'$, $\underline{I}'$ and $\underline{K}'$ are given by

$$\underline{H} = \pm\sqrt{H^2 - M^2} \quad \underline{I} = \pm\frac{1}{3}\{H^2 - M^2\}^{1/2}, \quad \underline{K} = \pm\frac{1}{3}\{H^2 - M^2\}^{1/2},$$

$$\underline{J} = \underline{J}' = \underline{H}' = \underline{I}' = \underline{K}' = 0.$$

Corollary: In a five-dimensional C-reducible Finsler space, the main scalars *H* satisfies the condition: $H > M$ or $H < -M$.

Now, suppose the torsion vector C_i of F^5 is tangent to its hypersurface F^4, then from (2.7), $t = 0$. Therefore from (3.3), we get

$$\underline{X} = 0, \quad \underline{Z} = 0, \quad \underline{T} = 0, \quad \underline{Y} = 0, \quad \underline{S} = 0, \quad \underline{U} = 0, \quad \underline{V} = 0, \quad \underline{W} = 0, \quad \underline{M} = 0.$$

Thus, we get

Theorem: If the torsion vector C_i of a five-dimensional C-reducible Finsler space is tangent to its hypersurface, then $M_{\alpha\beta}$ and *M* defined by (7) vanish.

Matsumoto [3] showed an important result for connection of the hypersurface that if $M_{\alpha\beta} = 0$, then the induced and intrinsic connections of the hypersurface coincide. This leads to:

Corollary: If the torsion vector C_i of a five-dimensional C-reducible Finsler space is tangent to its hypersurface; then the induced connection of the hypersurface coincides with its intrinsic connection.

Now, if the torsion vector C_i of F^5 is normal to its hypersurface F^4, then from (17), *a* = *b* = *d* = 0. Therefore from (44), we get that all the main scalars $\underline{H}$, $\underline{I}$, $\underline{K}$, $\underline{J}$, $\underline{J}'$, $\underline{H}'$, $\underline{I}'$, $\underline{K}'$ of F^4 vanish, which is not possible.

Thus, we have

Theorem: The torsion vector C_i of a five-dimensional C-reducible Finsler space is not normal to its hypersurface.

References

1. Gauree Shanker, G. C. Chaubey and Vinay Pandey, *International Electronic J. Pure and Applied Math*., 5 (2012), 69-78.
2. M. Matsumoto and R. Miron, On an invariant theory of the Finsler spaces, Period. Math. Hunger, 8 (1977), 73-82.
3. M. Matsumoto, The induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry, *J. Math. Kyoto Univ*., 25-1 (1985), 107-144.
4. M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, Kaiseisha press, Saikawa, Otsu, 520 Japan (1986).

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